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LINE-GEOMETRIC REPRESENTATIONS FOR FUNCTIONS OF A COMPLEX VARIABLE.*

BY

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INTRODUCTION

The usefulness of the usual method of analytic geometry, for visualizing a functional relation between two *real* variables, is too well known to call for further comment. A second method, not quite so familiar, has also been used extensively, especially in projective geometry. It consists in constructing scales on two straight lines, one for the independent and one for the dependent variable, joining corresponding points of the two scales by straight lines, and studying the one-parameter family of straight lines obtained in this way.

This second method can be utilized also for the purpose of visualizing a functional relation between two *complex* variables. If $z = x + iy$, and $w = u + iv$ are two complex variables, and if w is a function of z, we may proceed as follows. Choose any two planes of ordinary space as the planes of the two complex variables, and join by straight lines the points which corspond to each other in the functional relation. The result will be a two-parameter family of lines, a congruence, and this congruence may be regarded as a geometric image of the functional relation. It should be mentioned immediately, however, that the properties of this congruence depend, not merely on the properties of the function $w = F(z)$, but also upon the relative position of the two planes. We shall, in this paper, consider only the case which suggests itself most naturally, namely the case of two parallel planes. Only in this case will all of the points which represent finite values of z be geometrically distinct from those which represent finite values of w.

Weierstrass, in a very brief and incidental manner, mentions the possibility of such a line-geometric image of a functional relation.† But he did not investigate the properties of the resulting congruences; in fact he disposes of the whole matter in a very few words. There exists, however, an extensive paper on this subject by M. J. Van Uven.[*] The work of Van Uven is devoted entirely to the discussion of special cases, such as $w = z^n$, $w = z^{-n}$, etc.; these are investigated in most thorough fashion, but the general questions are not attacked. A very interesting paper by Emch[†] makes use of a method which is applicable only to the case of a linear function. But, in spite of this limitation, Emch's paper is important, because in his paper the planes of the two complex variables are not parallel. Thus we are in a position to see, by comparing Emch's results with our own, that the mutual position of the two planes is an important feature of the theory. In fact we shall see that, for parallel planes, related to each other as described in Article 1, the congruence always has imaginary cones as focal surfaces, while Emch obtains real non-developable focal surfaces for the congruence which corresponds to the case which he has studied.

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* Presented to the Society, December 28, 1918, and April 4, 1919.

† K. Weierstrass, *Mathematische Werke. Vorlesungen über die Theorie der Abel'schen Transcendenten,* Chapter 16, p. 323.

In Article 2 we shall present a second method of representing $w = F(z)$ by means of a congruence. This method consists in interpreting both variables as points of the same Riemann sphere, and then joining corresponding points by straight lines. The focal surfaces of the congruences obtained by this method are always real and possess some very striking properties. The possibility of such a representation seems to have occurred to Study,[‡] but he did not develop the idea in detail. The method of Coolidge,[§] which makes use of congruences in non-euclidean (elliptic) space, is different from any of these and far less elementary.

1. THE METHOD OF PARALLEL PLANES

Let us write

$$(1)\qquad x + iy = z, \qquad x - iy = z_0, \qquad u + iv = w, \qquad u - iv = w_0,$$

and let us assume that w is a function of z, say

$$(2)\qquad w = F(z),$$

so that

$$(3)\qquad w_0 = F_0(z_0),$$

where F_0 denotes the function conjugate to F.

We interpret $z = x + iy$ in the usual way as a point P_z of the $\xi\eta$ plane, so that the cartesian coördinates of P_z will be

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[*] M. J. Van Uven, *Algebraische Strahlencongruenzen und verwandte komplexe Ebenen als Schmitte derselben.* Verhandelingen der Konniklijke Akademie van Wetenschappen te Amsterdam (Eerste Sectie), vol. 10 (1910), pp. 1–527.

[†] A. Emch, *On the rectilinear congruence realizing a circular transformation of one plane into another*, Annals of Mathematics, ser. 2, vol. 13 (1911–12), p. 155.

Study, *Über Nicht-Euklidische und Linien-Geometrie*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 11 (1902), p. 329.

[§] J. L. Coolidge, *Les congruences isotropes qui servent à représenter les fonctions d'une variable complexe*, Atti di Torino, vol. 38.

$$\xi_1 = x, \qquad \eta_1 = y, \qquad \zeta_1 = 0.$$

We place the plane of the complex variable w parallel to that of z, in such a way that the origin of the w-plane shall be on the ζ-axis at the point $(0, 0, 1)$, the u- and v-axes being parallel to the x- and y-axes respectively. Then the cartesian coördinates of P_w will be

$$\xi_2 = u, \qquad \eta_2 = v, \qquad \zeta_2 = 1.$$

Let the *homogeneous* cartesian coördinates of P_z and P_w be $(\lambda_1, \cdots \lambda_4)$. and $(\mu_1, \cdots \mu_4)$ respectively. If we express x, y, u, and v in terms of z, z_0, w, w_0, by means of (1), we may write

$$(4) \qquad \begin{aligned} \lambda_1 &= \frac{1}{2}(z + z_0), & \lambda_2 &= \frac{1}{2i}(z - z_0), & \lambda_3 &= 0, & \lambda_4 &= 1, \\ \mu_1 &= \frac{1}{2}(w + w_0), & \mu_2 &= \frac{1}{2i}(w - w_0), & \mu_3 &= 1, & \mu_4 &= 1, \end{aligned}$$

where, of course, only the ratios $\lambda_i : \lambda_k$ and $\mu_i : \mu_k$ are of significance.

In order to study the congruence, which is obtained when we join corresponding points P_z and P_w, we make use of a system of linear homogeneous differential equations satisfied by each of the four pairs of functions (λ_k, μ_k). This system of partial differential equations will be composed of two equations of the first order, and of two equations of the second order. The second order equations are obtained very easily. Obviously the equations

$$(5) \qquad \frac{\partial^2 \lambda}{\partial z^2} = 0, \qquad \frac{\partial^2 \lambda}{\partial z_0^2} = 0, \qquad \frac{\partial^2 \mu}{\partial w^2} = 0, \qquad \frac{\partial^2 \mu}{\partial w_0^2} = 0$$

are satisfied by each of the eight functions λ_k, μ_k. The latter two assume the form

$$(6) \qquad \frac{\partial^2 \mu}{\partial z^2} - \frac{w''}{w'}\frac{\partial \mu}{\partial z} = 0, \qquad \frac{\partial^2 \mu}{\partial z_0} - \frac{w_0''}{w_0'}\frac{\partial \mu}{\partial z_0} = 0,$$

if we introduce z and z_0 as independent variables, and if we write

$$w' = \frac{dw}{dz}, \qquad w'' = \frac{d^2 w}{dz^2}, \qquad w_0 = \frac{dw_0}{dz_0}, \qquad w'' = \frac{d^2 w_0}{dz_0^2}.$$

In this connection the following remark is essential. If we maintain our original agreement that z and z_0 are conjugate complex variables, that is, that x and y are real, we can not treat z and z_0 as independent complex variables. Consequently we generalize our problem, at this stage, by thinking of x and y as independent complex variables. In most of the applications, however, we return to our original hypothesis, that x and y are real.

Direct substitution of the four pairs of functions (λ_k, μ_k) enables us to verify that each of these pairs also satisfies the two first order equations

$$(7)\qquad w'\frac{\partial\lambda}{\partial z}-\frac{\partial\mu}{\partial z}=0,\qquad w_0'\frac{\partial\lambda}{\partial z_0}-\frac{\partial\mu}{\partial z_0}=0.$$

Let us put

$$(8)\qquad \bar{\lambda}=w'\lambda-\mu,\qquad \bar{\mu}=w_0'\lambda-\mu,$$

whence

$$(9)\qquad (w'-w_0')\lambda=\bar{\lambda}-\bar{\mu},\qquad (w'-w_0')\mu=w_0'\bar{\lambda}-w'\bar{\mu}.$$

Then we may write, in place of (7),

$$(10)\qquad \frac{\partial\bar{\lambda}}{\partial z}=\frac{w''}{w'-w_0'}(\bar{\lambda}-\bar{\mu}),\qquad \frac{\partial\bar{\mu}}{\partial z_0}=\frac{w_0''}{w'-w_0'}(\bar{\lambda}-\bar{\mu}),$$

if we assume $w'-w_0'\neq 0$. The case thus excluded has little interest, since it corresponds to the case when w' is a real constant, so that $w=kz+b$ (k real). In this case the congruence degenerates into a bundle of lines.

We proceed to interpret equations (10). Under the assumption

$$w'-w_0'\neq 0,$$

equations (8) give rise to two distinct points, $P_{\bar{\lambda}}$ and $P_{\bar{\mu}}$, on the line $P_z P_w$. Let P_z move along one of the null lines $z_0=$ const. of the xy-plane. Then the line $P_z.P_w$ generates an imaginary ruled surface of the congruence. The point $P_{\bar{\lambda}}$ describes a curve on this ruled surface whose tangent is obtained by joining $P_{\bar{\lambda}}$ to the point whose coördinates are $\partial\bar{\lambda}_k/\partial z$. According to (10) this tangent coincides with the line $P_{\bar{\lambda}}P_{\bar{\mu}}$, that is, with the line $P_z P_w$ itself. In other words, this ruled surface is a developable, and the locus of $P_{\bar{\lambda}}$ is its cuspidal edge. Moreover, since this developable contains a straight line (a null-line of the $\xi\eta$-plane), the cuspidal edge of the developable must be a plane curve, a fact which we shall also verify later analytically.

Thus, *the imaginary ruled surfaces of the congruence, which correspond to the two families of null lines of the* $\xi\eta$*-plane, constitute the developables of the congruence. Each of these developables touches one of the sheets of the focal surface along a plane curve.*

We can simplify the form of equations (10) by putting

$$(11)\qquad f=\frac{\bar{\lambda}}{w'-w_0'},\qquad g=\frac{\bar{\mu}}{w'-w_0'}.$$

They then assume the form

$$(12)\qquad \frac{\partial f}{\partial z}=mg,\qquad \frac{\partial g}{\partial z_0}=nf,$$

where

$$(13)\qquad m=\frac{-w''}{w'-w_0'},\qquad n=\frac{+w_0''}{w'-w_0'}.$$

Since we are using homogeneous coördinates, f and g will represent the same points as $\bar{\lambda}$ and $\bar{\mu}$ respectively, namely the two *foci* of a line of the congruence, that is, the two points in which $P_z P_w$ touches the two sheets of the focal surface.

By means of (9) and (11), we find

$$\lambda = f - g, \qquad \mu = w_0' f - w' g. \tag{14}$$

Equations (12) are the two first order differential equations, in the canonical form which is most convenient for the application of the general theory of congruences. The two second order equations may be obtained from (5) and (6) by introducing the variables f and g by means of (14). Two of the equations obtained in this way are

$$\frac{\partial^2 f}{\partial z_0^2} = \frac{\partial n}{\partial z_0} f + n \frac{\partial f}{\partial z_0}, \qquad \frac{\partial^2 g}{\partial z^2} = \frac{\partial m}{\partial z} g + m \frac{\partial g}{\partial z}; \tag{15}$$

the other two are consequences of (12) and (15).

Thus we have found that *our four pairs of functions* (f_k, g_k) *are solutions of a system of differential equations of the form**

$$\begin{aligned} &\frac{\partial f}{\partial z} = mg, \qquad \frac{\partial g}{\partial z_0} = nf, \\ &\frac{\partial^2 f}{\partial z_0^2} = af + bg + c\frac{\partial f}{\partial z_0} + d\frac{\partial g}{\partial z}, \\ &\frac{\partial^2 g}{\partial z^2} = a'f + b'g + c'\frac{\partial f}{\partial z_0} + d'\frac{\partial g}{\partial z}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} &m = \frac{-w''}{w' - w_0'}, \qquad n = \frac{+w_0''}{w' - w_0'}, \\ &a = \frac{\partial n}{\partial z_0}, \qquad b = 0, \qquad c = n, \qquad d = 0, \\ &a' = 0, \qquad b' = \frac{\partial m}{\partial z}, \qquad c' = 0, \qquad d' = m. \end{aligned} \tag{17}$$

In general, the locus of P_f, one of the sheets of the focal locus is a surface. It degenerates into a curve, if and only if m is equal to zero. This is an obvious consequence of (16), since f will then be a function of z_0 alone. But m can vanish identically, only if w is an integral linear function of z, and in that case n also will vanish, and therefore all of the coefficients of (16) will be equal to zero. To investigate the nature of the focal locus in this case, we make use of equations (4) and (8), after making the substitutions

$$w = \alpha z + \beta, \qquad w_0 = \alpha_0 z_0 + \beta_0,$$

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* This is the form of the system upon which I have based the general projective theory of congruences. See *Sur la théorie générale des congruences*, Mémoires publies par la Classe des Sciences de l'Academie Royale de Belgique; 2 série, vol. 3 (1911). Hereafter quoted as *Brussels Paper*.

where α and β, α_0 and β_0, are constants, and where we shall assume $\alpha \neq \alpha_0$ so as to avoid the exceptional case $w' = w'_0$ whose significance has been pointed out already. A simple calculation shows that the focal locus consists of two straight lines, one through each of the circular points of the $\xi\eta$-plane.

In all other cases, m and n are different from zero, and the focal locus is composed of two surfaces. Equations (17) show that c' and d are equal to zero. Consequently *the focal surface of the congruence is composed of two developables.** Evidently the case $m = n = 0$ may be included in this statement, if we agree to regard a straight line as a degenerate case of a developable. Since we have, according to (15),

$$\frac{\partial^2 f}{\partial z_0} = \frac{\partial n}{\partial z_0} f + n \frac{\partial f}{\partial z_0}, \tag{18}$$

the curves on the sheet S_f of the focal surface, which correspond to the null lines z = const. of the $\xi\eta$-plane, are straight lines. We have seen already that the curves on S_f which correspond to the null-lines z_0 = const. are plane curves.

The first part of this theorem becomes quite obvious, if we remember that the two families of developables of the congruence must determine a conjugate system of curves on each sheet of the focal surface. In our case the focal surfaces are developable, and on a developable one of the component families of a conjugate system always consists of its generators.

If we put

$$h = \frac{\partial f}{\partial z_0} - nf, \tag{19}$$

and make use of (16) and (17), we find

$$\frac{\partial h}{\partial z} = 0, \qquad \frac{\partial h}{\partial z_0} = 0. \tag{20}$$

But clearly the point P_h, whose coördinates $(h_1, \cdots, h_4)$ are given by (19), is on that generator of the developable S_f which corresponds to a given constant value of z. According to (20), this point P_h is the same for all such generators; consequently *the focal surface S_f is a cone.* We may write

$$h = \frac{\partial f}{\partial z_0} - \frac{\partial g}{\partial z_0} = \frac{\partial \lambda}{\partial z_0}.$$

If we substitute the values (4) for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we find

$$h_1 : h_2 : h_3 : h_4 = i : -1 : 0 : 0. \tag{21}$$

Therefore, *the vertex of the focal cone S_f is one of the circular points of the $\xi\eta$-plane.*

The curves z = const. of S_f are the generators of the cone. The curves z_0 = const. of S_f are plane curves, a fact which we have mentioned before, but which may also be proved as follows. We have

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* Brussels Paper, p. 28.

$$\frac{\partial f}{\partial z} = mg, \qquad \frac{\partial^2 f}{\partial z^2} = \frac{\partial m}{\partial z} g + m \frac{\partial g}{\partial z},$$

$$\frac{\partial^3 f}{\partial z^3} = \left(\frac{\partial^2 m}{\partial z^2} + m \frac{\partial m}{\partial z} \right) g + \left(2 \frac{\partial m}{\partial z} + m^2 \right) \frac{\partial g}{\partial z}.$$

From these three equations, we may eliminate g and $\partial g/\partial z$. Thus we obtain the linear homogeneous differential equation of the third order for f;

$$(22) \quad \frac{\partial^3 f}{\partial z^3} - \left(m + 2 \frac{\partial \log m}{\partial z} \right) \frac{\partial^2 f}{\partial z^2} - \left[\frac{\partial^2 \log m}{\partial z^2} - \left(\frac{\partial \log m}{\partial z} \right)^2 \right] \frac{\partial f}{\partial z} = 0.$$

Since f_1, f_2, f_3, f_4 are solutions of this third order equation, the curves $z_0 = \text{const.}$ on S_f are indeed plane curves. Moreover equation (22) may serve for the purpose of discussing the projective properties of these curves, and therefore of the focal cone. But, unless we specialize the function $w = F(z)$, we need not expect to find any special properties of the focal cone beyond those already noted. In fact we shall show that this focal cone may be regarded as the most general analytic cone which has its vertex at $(i, -1, 0, 0)$.

To prove this we observe in the first place that the cone S_f is the envelope of the one-parameter family of planes

$$(23) \qquad x_1 + ix_2 + [z - F(z)]\, x_3 - zx_4 = 0,$$

all of which, of course, pass through the circular point I $(1, i, 0, 0)$, which is the vertex of the cone. Now the most general one-parameter family of planes through I may be represented by an equation of the form

$$(24) \qquad x_1 + ix_2 + \varphi(t) x_3 + \psi(t) x_4 = 0.$$

If $\psi(t)$ does not reduce to a constant, we may put $\psi(t) = -z$, and then determine $F(z)$ by means of the equation

$$z - F(z) = \varphi(t),$$

giving

$$F(z) = -\varphi(t) - \psi(t).$$

Thus, unless $\psi(t) = \text{const.}$, any family of form (24) is included in (23). If $\psi(t)$ is a constant, $\varphi(t)$ can not also be a constant if (24) is to represent a one-parameter family of planes. In the case $\psi(t) = \text{const.}$, (24) therefore represents a pencil of planes whose axis is a straight line in the xy-plane passing through I. This case only presents itself when z regarded as a function of w is a constant.

When the focal cone S_f is given, the second focal cone S_g is uniquely determined. It is the envelope of the one-parameter family of planes

$$x_1 - ix_2 + [z_0 - F_0(z_0)]\, x_3 - z_0 x_4 = 0 \tag{25}$$

which are conjugate to (23).

Of course, the equations (23) and (25), regarded as simultaneous, are the equations of the two-parameter family of lines which constitute our congruence. In this form they are especially adapted to answer certain questions which arise naturally, especially when $F(z)$ is algebraic, as to the number of lines of the congruence which pass through a given point, or lie in a given plane, etc.

We may summarize our principal results as follows.

1. *If* $w = F(z) =$ *const., the congruence reduces to a bundle of lines whose vertex is the point* $w =$ *const. of the w-plane. Similarly if* $z = F^{-1}(w) =$ *const.*

2. *If* $w = kz + \beta$, *where k and* β *are constants and where k is real, the congruence still reduces to a bundle of lines, but its vertex does not lie in the plane of either complex variable.*

3. *If* $w = \alpha z + \beta$, *where* α *and* β *are constants and where* α *is not real, the focal locus of the congruence is composed of two straight lines, one through each of the two circular points which the planes of z and w have in common.*

4. *In all other cases, the focal locus is composed of two cones whose vertices are the two circular points which the planes of z and w have in common. The developables of the congruence correspond to the null-lines of the two planes. Each of these developables is the locus of the tangents of a plane curve on one of the focal cones, and touches the other focal cone along a generator. The planes of the plane curves, thus determined on each of the focal cones, are the tangent planes of the other focal cone.*

Interesting as these results are, they do not help us to visualize the properties of the function $w = F(z)$, except in the simple cases 1 and 2, in which no such aid is needed. This is due to the fact that, in the general case, the developables and focal surfaces of the congruence are imaginary. In this respect the method of representation by means of the Riemann sphere (see Art. 2) is far superior. Nevertheless it is easy to utilize the method of parallel planes for furnishing real images of the functional relation. Thus every real curve of the z-plane determines a real ruled surface of the congruence which intersects the w-plane in a corresponding curve. These ruled surfaces, especially those which correspond to the lines $x =$ const. and $y =$ const., help us to understand the conformal correspondence between the two planes.* Moreover, certain loci connected with these ruled surfaces give rise to independent methods of defining the congruence.

We shall not, at present, discuss any questions of this general nature. We shall, however, indicate a method which enables us to replace the imaginary focal surfaces by certain real surfaces very closely connected with them. The foci, P_f and P_g, of a line of the congruence are conjugate complex points of the line. Consequently the harmonic conjugate of P_z with respect to P_f and P_g will be a real point. We shall denote this real point by P_k, and the locus of P_k will be a surface S_k, whose properties we propose to investigate. Similarly we shall study the surface S_v, locus of P_v, where P_v is the harmonic conjugate of P_w with respect to P_f and P_g.

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* Van Uven's paper is largely devoted to the study of such ruled surfaces in the various special cases which he discusses.

It follows from (14) that the homogeneous coördinates of P_k are given by

$$k = f + g. \tag{26}$$

We find

$$\frac{\partial k}{\partial z} = mg + \frac{\partial g}{\partial z}, \qquad \frac{\partial k}{\partial z_0} = nf + \frac{\partial f}{\partial z_0}, \tag{27}$$

and

(28
$$\frac{\partial^2 k}{\partial z^2} = 2m_z g + 2mg_z, \qquad \frac{\partial^2 k}{\partial z_0^2} = 2n_{z_0} f + 2nf_{z_0},$$
$$\frac{\partial^2 k}{\partial z \partial z_0} = (m_{z_0} + mn) g + (n_z + mn) f,$$

where we have written

$$m_z = \frac{\partial m}{\partial z}, \qquad n_z = \frac{\partial n}{\partial z}, \qquad \text{etc.}$$

But, according to (17), we have

$$m_{z_0} = mn, \qquad n_z = mn, \tag{29}$$

so that we find

$$\frac{\partial^2 k}{\partial z \partial z_0} = 2mnk. \tag{30}$$

Consequently, *the curves on S_k, which correspond to the null-lines of the plane, form a conjugate system with equal Laplace-Darboux invariants.**

Moreover, *the curves of this conjugate system S_k are plane curves.*† For we have

$$\frac{\partial^3 k}{\partial z^3} = 2(m_{zz} + mm_z) g + 2(2m_z + m^2) g_z. \tag{31}$$

From this equation, and the equations for $\partial k/\partial z$ and $\partial^2 k/\partial z^2$, we may eliminate g and g_z, obtaining a linear homogeneous differential equation of the third order for the curve $z_0 =$ const. on S_k, viz.;

$$(m^2 - m_z)\frac{\partial^3 k}{\partial z^3} + (m_{zz} - mm_z - m^3)\frac{\partial^2 k}{\partial z^2} - 2(mm_{zz} - 2m_z^2)\frac{\partial k}{\partial z} = 0. \tag{32}$$

Only if $m_z = m^2$ will this plane curve reduce to a straight line. But we find, from (13),

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* G. Darboux, *Théorie des surfaces*, vol. 2, p. 27.

† This remark was first made by one of my students, Dr. Cyril Nelson.

$$(33) \qquad m_z = m^2 + \frac{w'''}{w''} m, \qquad n_{z_0} = n^2 + \frac{w_0'''}{w_0''} n.$$

Therefore, if $m_z = m^2$, we must have either $m = 0$ or $w''' = 0$. The former case corresponds to $w = \alpha z + \beta$, where α and β are constants, and in this case S_k degenerates into a plane. If $m_z = m^2$, $m \neq 0$, we have $w''' = 0$, or $w = \alpha z^2 + \beta z + \gamma$, where α, β, and γ are constants, and $n_{z_0} = n^2$. Consequently we find, in this case,

$$k_{zz} = 2mk_z, \qquad k_{z_0 z_0} = 2nk_{z_0}, \qquad k_{zz_0} = 2mnk,$$

so that, in this case also, S_k degenerates into a plane,* and the curves $z =$ const. and $z_0 =$ const. in this plane are straight lines.

If we discard these cases, each of the curves $z =$ const. and $z_0 =$ const. of S_k, which intersect at P_k, determine a plane. The plane of the curve $z_0 =$ const. is determined by

$$k = f + g, \qquad \frac{\partial k}{\partial z} = mg + g_z, \qquad \frac{\partial^2 k}{\partial z^2} = 2m_z g + 2mg_z;$$

that of the curve $z =$ const. by

$$k = f + g, \qquad \frac{\partial k}{\partial z_0} = nf + f_{z_0}, \qquad \frac{\partial^2 k}{\partial z_0^2} = 2n_{z_0} f + 2nf_{z_0}.$$

Of course $k = f + g$ is a point of both planes. Since we may write

$$(34) \qquad k_{zz} - 2mk_z = 2(m_z - m^2)g$$

and

$$(35) \qquad k_{z_0 z_0} - 2nk_{z_0} - 2(n_{z_0} - n^2)k = -2(n_{z_0} - n^2)g,$$

we see that the point P_g is common to both planes, and therefore the same thing is true of P_f.

In other words; *the lines of our congruence are defined completely if the real surface S_k and the conjugate system of plane curves on S_k is known. They are the lines of intersection of the planes of these curves. Our congruence is the so-called axis congruence of this conjugate system on S_k.*† It is clear, moreover, that the planes of the curves on S_k which form the conjugate system are the tangent planes of the two focal cones.

In order to obtain a complete projective theory of the surface S_k, we must find the two second order partial differential equations which k_1, k_2, k_3, k_4 will satisfy. One of these is (30). The other one may be obtained from (34) and (35) by eliminating g. It then appears in the form

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* E. J. Wilczynski, *One-parameter family of nets of plane curves.* These Transactions, vol. 12 (1911), p. 474.

† E. J. Wilczynski, *The general theory of congruences,* These Transactions vol. 16, (1915), pp. 312–317.

$$(36)\quad \begin{aligned}(n_{z_0} - n^2)(k_{zz} - 2mk_z) + (m_z - m^2)(k_{z_0z_0} - 2nk_{z_0})&\\ = 2(m_z - m^2)(n_{z_0} - n^2)k&.\end{aligned}$$

But, on account of (13) and (33), this becomes

$$(37)\quad w_0''' k_{zz} - w''' k_{z_0z_0} + \frac{2w'' w_0'''}{w' - w_0'} k_z + \frac{2w''' w_0''}{w' - w_0'} k_{z_0} + \frac{2w''' w_0'''}{w' - w_0'} k = 0,$$

where we shall assume $w' - w_0' \neq 0$, $w''' \neq 0$.

This equation shows that the coefficients of the second fundamental form of S_k, with respect to the parameters z and z_0, are proportional to

$$w''', \qquad 0, \qquad w_0''',$$

so that the differential equation of the asymptotic lines of S_k is

$$w''' dz^2 + w_0''' dz_0^2 = 0.$$

Consequently *the conjugate system formed on S_k by the curves z = const. and z_0 = const. is an isothermally conjugate system.* The asymptotic lines of S_k are obtained by equating to constants the variables

$$R = \frac{1}{2}\left[\int \sqrt{w'''}\, dz + i\int \sqrt{w_0'''}\, dz_0\right], \qquad S = \frac{1}{2}\left[\int \sqrt{w'''}\, dz - i\int \sqrt{w_0'''}\, dz_0\right].$$

These variables are not real, but we shall show that they differ from real variables only by constant imaginary factors. For, if we put

$$\int \sqrt{w'''}\, dz = \lambda + i\mu, \qquad \int \sqrt{w_0'''}\, dz_0 = \lambda - i\mu,$$

where λ and μ are real, we find

$$R = \frac{1+i}{2}(\lambda + \mu), \qquad S = \frac{1-i}{2}(\lambda - \mu).$$

Consequently the variables

$$r = \frac{1-i}{\sqrt{2}} R, \qquad s = -\frac{1+i}{\sqrt{2}} S$$

are real and constant along the asymptotic lines of S_k. We find

$$\begin{aligned} p &= r + is\\ &= \frac{1}{2\sqrt{2}}\left[(1-i)\int \sqrt{w'''}\, dz + (1+i)\int \sqrt{w_0'''}\, dz_0\right.\\ &\qquad\left. - i(1+i)\int \sqrt{w'''}\, dz - (1+i)\int \sqrt{w_0'''}\, dz_0\right],\end{aligned}$$

or

$$p = \frac{1-i}{\sqrt{2}} \int \sqrt{w'''}\, dz, \qquad p_0 = \frac{1+i}{\sqrt{2}} \int \sqrt{w_0'''}\, dz_0, \tag{38}$$

or, if we prefer,

$$p = \int \sqrt{iw'''}\, dz, \qquad p_0 = \int \sqrt{-iw_0'''}\, dz_0.$$

Since r and s are the components of the complex variable $p = r + is$, which is a function of $z = x + iy$, we see that *the asymptotic lines of S_k correspond to an orthogonal isothermal system of curves in the z-plane, namely to the curves obtained by equating to constants the real and the imaginary parts of the function*

$$p = \frac{1-i}{\sqrt{2}} \int \sqrt{w'''}\, dz.$$

Thus the asymptotic curves on S_k are real, and S_k is a surface of negative curvature.

If we introduce p and p_0 as independent variables, the two differential equations for k assume the form

$$\begin{gathered} \frac{\partial^2 k}{\partial p^2} + \frac{\partial^2 k}{\partial p_0^2} + \frac{1+i}{\sqrt{2}} \frac{1}{\sqrt{w'''}} \left[\frac{2w''}{w' - w_0'} + \frac{1}{2} \frac{w^{(4)}}{w'''} \right] \frac{\partial k}{\partial p} \\ - \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{w_0'''}} \left[\frac{2w_0''}{w' - w_0'} - \frac{1}{2} \frac{w_0^{(4)}}{w_0'''} \right] \frac{\partial k}{\partial p_0} + \frac{2i}{w' - w_0'} k = 0, \\ \frac{\partial^2 k}{\partial p \partial p_0} + \frac{2w'' w_0'' k}{(w' - w_0')^2 \sqrt{w''' w_0'''}} = 0. \end{gathered} \tag{39}$$

Since $p = r + is$, $p_0 = r - is$, we obtain from this the differential equations of S_k referred to its asymptotic lines, namely

$$\begin{aligned} \frac{\partial^2 k}{\partial r^2} + \frac{1}{2} \frac{W_r}{W} \frac{\partial k}{\partial r} - \frac{1}{2} \frac{W_s}{W} \frac{\partial k}{\partial s} + \left[\frac{4w'' w_0''}{W} + \frac{2i}{w' - w_0'} \right] k = 0, \\ \frac{\partial^2 k}{\partial s^2} - \frac{1}{2} \frac{W_r}{W} \frac{\partial k}{\partial r} + \frac{1}{2} \frac{W_s}{W} \frac{\partial k}{\partial s} + \left[\frac{4w'' w_0''}{W} - \frac{2i}{w' - w_0'} \right] k = 0, \end{aligned} \tag{40}$$

where

$$W = (w' - w_0')^2 \sqrt{w''' w_0'''}. \tag{41}$$

We have already observed that we obtain a second real surface S_v as the locus of P_v where P_v is the harmonic conjugate of P_w with respect to P_f and P_g. Since the homogeneous coördinates of P_w are given by $\mu = w_0' f - w' g$, those of P_v will be

$$v = w_0' f + w' g. \tag{42}$$

We find

$$\frac{\partial v}{\partial z} = (w_0' m + w'') g + w' g_z,$$

$$\frac{\partial^2 v}{\partial z \partial z_0} = (w_0'' m + w_0' mn + w' mn) g + (w'' n + w_0' mn + w' mn) f,$$

which reduces to

$$\frac{\partial^2 v}{\partial z \partial z_0} = 2mnv. \tag{43}$$

Of course the properties of S_v, in relation to our congruence, are essentially the same as those of S_k, but equation (43) shows further that S_k and S_v *satisfy the same Moutard equation.*

2. THE METHOD OF THE RIEMANN SPHERE

We now pass to the second method mentioned in the introduction. We have, as before,

$$w = u + iv = F(z) = F(x + iy).$$

We use the $\xi\eta$-plane as the plane of both of the complex variables. Let P_z and P_w be two points of the $\xi\eta$-plane which represent a value of z and the corresponding value of w, and let P_1 and P_2 be the two points obtained from P_z and P_w, upon the sphere of unit radius and center at the origin, by stereographic projection from the point $(0, 0, 1)$. The coördinates of P_1 and P_2 will be

$$\xi_1 = \frac{z + z_0}{zz_0 + 1}, \qquad \eta_1 = \frac{-i(z - z_0)}{zz_0 + 1}, \qquad \zeta_1 = \frac{zz_0 - 1}{zz_0 + 1},$$

and

$$\xi_2 = \frac{w + w_0}{ww_0 + 1}, \qquad \eta_2 = \frac{-i(w - w_0)}{ww_0 + 1}, \qquad \zeta_2 = \frac{ww_0 - 1}{ww_0 + 1},$$

respectively. We may therefore write for the homogeneous cartesian coordinates of P_1,

$$\lambda_1 = z + z_0, \qquad \lambda_2 = -i(z - z_0), \qquad \lambda_3 = zz_0 - 1, \qquad \lambda_4 = zz_0 + 1, \tag{44}$$

and for those of P_2,

$$\mu_1 = w + w_0, \quad \mu_2 = -i(w - w_0), \quad \mu_3 = ww_0 - 1, \quad \mu_4 = ww_0 + 1, \tag{45}$$

where, of course,

$$w = F(z), \qquad w_0 = F_0(z_0).$$

We propose to study the congruence generated by the line $P_1 P_2$ when $z = x + iy$ ranges over the domain in which the function $F(z)$ is defined. In order to apply the general theory of congruences we must find two linear partial differential equations of the first order, and two of the second order which are satisfied by the four pairs of functions (λ_k, μ_k). Since λ_k is linear in z, and μ_k is linear in w, the second order equations may be taken in the form

$$\frac{\partial^2 \lambda}{\partial z^2} = 0, \qquad \frac{\partial^2 \mu}{\partial z^2} - \frac{w''}{w'}\frac{\partial \mu}{\partial z} = 0, \tag{46}$$

the latter being derived from

$$\frac{\partial^2 \mu}{\partial w^2} = 0$$

by introducing z as independent variable. Of course we have also

$$\frac{\partial^2 \lambda}{\partial z_0^2} = 0, \qquad \frac{\partial^2 \mu}{\partial z_0^2} - \frac{w_0''}{w_0'} \frac{\partial \mu}{\partial z_0} = 0, \tag{47}$$

but these are consequences of (46) and the two first order equations which are still to be found. To obtain these we use the method of undetermined coefficients. We write

$$\begin{aligned} \frac{\partial \lambda}{\partial z_0} &= a_{11} \frac{\partial \lambda}{\partial z} + a_{12} \frac{\partial \mu}{\partial z} + b_{11} \lambda + b_{12} \mu, \\ \frac{\partial \mu}{\partial z_0} &= a_{21} \frac{\partial \lambda}{\partial z} + a_{22} \frac{\partial \mu}{\partial z} + b_{21} \lambda + b_{22} \mu, \end{aligned} \tag{48}$$

and determine the coefficients a_{ik} and b_{ik} by demanding that each of the equations (48) shall be satisfied when we put

$$\lambda = \lambda_k, \qquad \mu = \mu_k, \qquad (k = 1, 2, 3, 4).$$

We find

$$\begin{aligned} &a_{11} = 0, \qquad a_{12} = -\frac{z - w}{z_0 - w_0} \frac{dz}{dw}, \qquad b_{11} = -b_{12} = \frac{1}{z_0 - w_0}, \\ &a_{21} = -\frac{z - w}{z_0 - w_0} \frac{dw_0}{dz_0}, \qquad a_{22} = 0, \qquad b_{21} = -b_{22} = \frac{\frac{dw_0}{dz_0}}{z_0 - w_0}. \end{aligned} \tag{49}$$

Thus, the desired system of partial differential equations is the following:

$$\begin{aligned} \frac{\partial \lambda}{\partial z_0} &= -\frac{z - w}{z_0 - w_0} \frac{dz}{dw} \frac{\partial \mu}{\partial z} + \frac{1}{z_0 - w_0} (\lambda - \mu), \\ \frac{\partial \mu}{\partial z_0} &= -\frac{z - w}{z_0 - w_0} \frac{dw_0}{dz_0} \frac{\partial \lambda}{\partial z} + \frac{\frac{dw_0}{dz_0}}{z_0 - w_0} (\lambda - \mu), \\ \frac{\partial^2 \lambda}{\partial z^2} &= 0, \qquad \frac{\partial^2 \mu}{\partial z^2} - \frac{w''}{w'} \frac{\partial \mu}{\partial z} = 0. \end{aligned} \tag{50}$$

Equations (46) show that *the ruled surfaces* $z_0 =$ *const. of the congruence have two straight-line directrices*. That this must be so is, of course, evident. For, on account of the conformal character of the correspondence between z and w, and since the stereographic projection is also conformal, z and w will both describe null-lines of the $\xi\eta$-plane if either of them does, and then the points P_1 and P_2 will simultaneously describe null-lines (imaginary generators) of the sphere.

The foci of the line $P_1 P_2$, that is the two points where the line touches the focal surface of the congruence, may be obtained by means of general methods familiar to geometers. As applied to a system of form (48), these methods show that the homogeneous coördinates of the foci are the factors of the quadratic covariant

$$a_{21}\lambda^2 - (a_{11} - a_{22})\lambda\mu - a_{12}\mu^2 .$$

If we use (49), we find that, in our case, the foci are given by

$$\bar{\lambda} = \mu + \sqrt{w' w_0'}\,\lambda , \qquad \bar{\mu} = \mu - \sqrt{w' w_0'}\,\lambda , \tag{51}$$

where

$$w' = \frac{dw}{dz} = F'(z) , \qquad w_0' = \frac{dw_0}{dz_0} = F_0'(z_0) .$$

Since $w' w_0'$ is positive, and since λ_k and μ_k are real, we see that *the focal surfaces of this congruence are always real.* Moreover, equations (51) show, that *the foci of every line of the congruence separate harmonically the two points, P_1 and P_2, in which this line intersects the Riemann sphere.* Finally we observe that *the two foci of a line are distinct except upon those lines of the congruence which correspond to a solution of the equation $F'(z) = 0$ or of $F'(z) = \infty$.*

Thus the two sheets of the focal surface can hang together only at those points for which $F'(z)$ becomes zero or infinite.

The developables of the congruence may be obtained, in the general case of a system of form (48), by equating to constants two independent solutions, $\varphi(z, z_0)$ and $\psi(z, z_0)$, of the partial differential equation

$$\left(\frac{\partial\theta}{\partial z_0}\right)^2 - (a_{11} + a_{22})\frac{\partial\theta}{\partial z}\frac{\partial\theta}{\partial z_0} + (a_{11}a_{22} - a_{12}a_{21})\left(\frac{\partial\theta}{\partial z}\right)^2 = 0 .^*$$

In our case, using the values of a_{ik} given by (49), it is easy to perform the integration. We find in this way the following result.

The developables of our congruence are the ruled surfaces given by $\alpha = const.$ and $\beta = const.$, where

$$\alpha + i\beta = t = i\int\frac{\sqrt{w'}\,dz}{z - w} , \qquad \alpha - i\beta = t_0 = -i\int\frac{\sqrt{w_0'}\,dz_0}{z_0 - w_0} . \tag{52}$$

Since α and β are the components of a complex variable t which is a function of z alone, *the developables of the congruence correspond to an orthogonal isothermal system of curves on the Riemann sphere.*

---

* Here, as well as in the determination of the focal surfaces, I am making use of some formulæ from a theory of one-parameter families of ruled surfaces which is, as yet, unpublished. The reader, who finds it difficult to prove these general formulæ, should follow the transformations mechanically. The final results enable him to verify, *a posteriori*, the correctness of our statements.

We shall have to transform equations (50) into an equivalent system for the functions $\bar{\lambda}$ and $\bar{\mu}$ of α and β, before we can apply the formulæ of the general theory of congruences to our particular case. This transformation may be accomplished as follows.

From (51), we find

$$\lambda = \frac{1}{2\sqrt{w' w_0'}}(\bar{\lambda} - \bar{\mu}), \qquad \mu = \tfrac{1}{2}(\bar{\lambda} + \bar{\mu}). \tag{53}$$

If we substitute these values in (50), we find

$$\begin{aligned} \frac{\partial\bar{\lambda}}{\partial z_0} &= -\sqrt{\frac{w_0'}{w'}}\frac{z - w}{z_0 - w_0}\frac{\partial\bar{\lambda}}{\partial z} + \frac{1}{2(z_0 - w_0)}\left(1 + \sqrt{\frac{w_0'}{w'}}\right)[\bar{\lambda} - \bar{\mu} \\ &\qquad - \sqrt{w' w_0'}(\bar{\lambda} + \bar{\mu})] + \frac{1}{4}\left[\frac{w_0''}{w_0'} + \frac{z - w}{z_0 - w_0}\sqrt{\frac{w_0'}{w'}}\frac{w''}{w'}\right](\bar{\lambda} - \bar{\mu}), \\ \frac{\partial\bar{\mu}}{\partial z_0} &= +\sqrt{\frac{w_0'}{w'}}\frac{z - w}{z_0 - w_0}\frac{\partial\bar{\mu}}{\partial z} - \frac{1}{2(z_0 - w_0)}\left(1 - \sqrt{\frac{w_0'}{w'}}\right)[\bar{\lambda} - \bar{\mu} \\ &\qquad - \sqrt{w' w_0'}(\bar{\lambda} + \bar{\mu})] - \frac{1}{4}\left[\frac{w_0''}{w_0'} - \frac{z - w}{z_0 - w_0}\sqrt{\frac{w_0'}{w'}}\frac{w''}{w'}\right](\bar{\lambda} - \bar{\mu}), \end{aligned} \tag{54}$$

and

$$\begin{aligned} &\frac{\partial^2\bar{\lambda}}{\partial z^2} - \frac{w''}{w'}\frac{\partial\bar{\lambda}}{\partial z} - \frac{1}{4}\{w, z\}(\bar{\lambda} - \bar{\mu}) = 0, \\ &\frac{\partial^2\bar{\mu}}{\partial z^2} - \frac{w''}{w'}\frac{\partial\bar{\mu}}{\partial z} + \frac{1}{4}\{w, z\}(\bar{\lambda} - \bar{\mu}) = 0, \end{aligned} \tag{55}$$

where $\{w, z\}$ is the Schwarzian derivative of w with respect to z, and where a definite determination must be chosen in every specific case of a branch of the function $\sqrt{w'}$; the value of $\sqrt{w_0'}$ will then be uniquely determined if we agree that $\sqrt{w' w_0'} = \sqrt{w'}\sqrt{w_0'}$ shall mean the positive value of this square root.

We continue the transformation by introducing the variables, t and t_0, defined by (52), in place of z and z_0. The result of this transformation on equations (54) is

$$\begin{aligned} -i\sqrt{w' w_0'}&\left(\frac{\partial\bar{\lambda}}{\partial t_0} - \frac{\partial\bar{\lambda}}{\partial t}\right) \\ &= +\left[\frac{1}{2}(\sqrt{w'} + \sqrt{w_0'})(1 - \sqrt{w' w_0'})\right. \\ &\qquad \left. + \frac{1}{4}\left\{(z_0 - w_0)\sqrt{w'}\frac{w_0''}{w_0'} + (z - w)\sqrt{w_0'}\frac{w''}{w'}\right\}\right]\bar{\lambda} \\ &\qquad - \left[\frac{1}{2}(\sqrt{w'} + \sqrt{w_0'})(1 + \sqrt{w' w_0'})\right. \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \frac{1}{4}\left\{ (z_0 - w_0)\sqrt{w'}\,\frac{w_0''}{w_0'} + (z - w)\sqrt{w_0'}\,\frac{w''}{w'} \right\}\Bigg]\overline{\mu}, \\
(56)\qquad & - i\sqrt{w' w_0'}\left(\frac{\partial\overline{\mu}}{\partial t_0} + \frac{\partial\overline{\mu}}{\partial t}\right) \\
&\qquad = -\left[\frac{1}{2}(\sqrt{w'} - \sqrt{w_0'})(1 - \sqrt{w' w_0'})\right. \\
&\qquad\qquad \left. + \frac{1}{4}\left\{ (z_0 - w_0)\sqrt{w'}\,\frac{w_0''}{w_0'} - (z - w)\sqrt{w_0'}\,\frac{w''}{w'} \right\}\right]\overline{\lambda} \\
&\qquad\qquad + \left[\frac{1}{2}(\sqrt{w'} - \sqrt{w_0'})(1 + \sqrt{w' w_0'})\right. \\
&\qquad\qquad \left. + \frac{1}{4}\left\{ (z_0 - w_0)\sqrt{w'}\,\frac{w_0''}{w_0'} - (z - w)\sqrt{w_0'}\,\frac{w''}{w'} \right\}\right]\overline{\mu}.
\end{aligned}
$$

It will not be necessary to apply this transformation to (55) just now; we shall do this presently however in a slightly modified form.

Since $t = \alpha + i\beta$, $t_0 = \alpha - i\beta$, we obtain, from (56), a system of the form

$$
(57)\qquad \frac{\partial\overline{\lambda}}{\partial\beta} = m'\,\overline{\lambda} + m''\,\overline{\mu}, \qquad \frac{\partial\overline{\mu}}{\partial\alpha} = n'\,\overline{\lambda} + n''\,\overline{\mu},
$$

where

$$
\begin{aligned}
& 4m' = 2\left(\frac{1}{\sqrt{w'}} + \frac{1}{\sqrt{w_0'}}\right)(1 - \sqrt{w' w_0'}) \\
&\qquad\qquad + (z_0 - w_0)\frac{w_0''}{(w_0')^{3/2}} + (z - w)\frac{w''}{(w')^{3/2}}, \\
& 4m'' = -2\left(\frac{1}{\sqrt{w'}} + \frac{1}{\sqrt{w_0'}}\right)(1 + \sqrt{w' w_0'}) \\
&\qquad\qquad - (z_0 - w_0)\frac{w_0''}{(w_0')^{3/2}} - (z - w)\frac{w''}{(w')^{3/2}}, \\
(58)\qquad & 4in' = 2\left(\frac{1}{\sqrt{w_0}} - \frac{1}{\sqrt{w'}}\right)(1 - \sqrt{w' w_0'}) \\
&\qquad\qquad + (z_0 - w_0)\frac{w_0''}{(w_0')^{3/2}} - (z - w)\frac{w''}{(w')^{3/2}}, \\
& 4in'' = -2\left(\frac{1}{\sqrt{w_0'}} - \frac{1}{\sqrt{w'}}\right)(1 + \sqrt{w' w_0'}) \\
&\qquad\qquad - (z_0 - w_0)\frac{w_0''}{(w_0')^{3/2}} + (z - w)\frac{w''}{(w')^{3/2}}.
\end{aligned}
$$

A transformation of the form

$$
\overline{\lambda} = \rho f, \qquad \overline{\mu} = \sigma g
$$

can always be made so as to transform a system of form (57) into one of the binomial form

$$
\frac{\partial f}{\partial\beta} = mg, \qquad \frac{\partial g}{\partial\alpha} = nf,
$$

by imposing the conditions

$$\frac{\partial \rho}{\partial \beta} = m' \rho, \qquad \frac{\partial \sigma}{\partial \alpha} = n'' \sigma$$

upon ρ and σ. We can satisfy these conditions in our case by putting

$$\rho = \sigma = \sqrt[4]{w' w_0'} \sqrt{(z - w)(z_0 - w_0)}, \tag{59}$$

so that we may even choose ρ and σ as positive quantities. Consequently the transformation

$$\begin{aligned} f &= \frac{\bar{\lambda}}{\sqrt[4]{w' w_0'} \sqrt{(z - w)(z_0 - w_0)}} = \frac{\sqrt[4]{w' w_0'}\, \lambda + \dfrac{\mu}{\sqrt[4]{w' w_0'}}}{\sqrt{(z - w)(z_0 - w_0)}}, \\ g &= \frac{\bar{\mu}}{\sqrt[4]{w' w_0} \sqrt{(z - w)(z_0 - w_0)}} = \frac{- \sqrt[4]{w' w_0'}\, \lambda + \dfrac{\mu}{\sqrt[4]{w' w_0'}}}{\sqrt{(z - w)(z_0 - w_0)}}, \end{aligned} \tag{60}$$

furnishes the first order equations of our system in their final form

$$f_\beta = mg, \qquad g_\alpha = nf, \tag{61}$$

where $f_\beta = \partial f/\partial \beta$, etc., and where m and n, which are real functions of α and β, are defined by

$$m + in = - \frac{1}{\sqrt{w'}} \left[1 + w' + \frac{1}{2}(z - w) \frac{w''}{w'} \right], \tag{62}$$

whence

$$m - in = - \frac{1}{\sqrt{w_0'}} \left[1 + w_0' + \frac{1}{2}(z_0 - w_0) \frac{w_0''}{w_0'} \right]. \tag{63}$$

By means of (60) we have four pairs of functions (f_k, g_k) corresponding to the four pairs (λ_k, μ_k). Since the coördinates, which we are using, are homogeneous the points P_f and P_g coincide with the foci $P_{\bar{\lambda}}$ and $P_{\bar{\mu}}$ respectively.

In order to complete the transformation of our second order equations, we first use (60) to express $\bar{\lambda}$ and $\bar{\mu}$ in terms of f and g. The result of the substitution of these values in (55) is

$$\begin{aligned} &\frac{\partial^2 f}{\partial z^2} - \frac{t''}{t'} \frac{\partial f}{\partial z} - \frac{1}{4} \left[\left(\frac{w''}{w'} - \frac{t''}{t'} \right)^2 + \frac{2w''}{z - w} \right] f + \frac{1}{4} \{ w, z \} g = 0, \\ &\frac{\partial^2 g}{\partial z^2} - \frac{t''}{t'} \frac{\partial g}{\partial z} + \frac{1}{4} \{ w, z \} f - \frac{1}{4} \left[\left(\frac{w''}{w'} - \frac{t''}{t'} \right)^2 + \frac{2w''}{z - w} \right] g = 0, \end{aligned} \tag{64}$$

whence

$$\begin{aligned} &\frac{\partial^2 f}{\partial t^2} - \frac{1}{4(t')^2} \left[\left(\frac{w''}{w'} - \frac{t''}{t'} \right)^2 + \frac{2w''}{z - w} \right) \Bigg] f + \frac{\{w, z\}}{4(t')^2} g = 0, \\ &\frac{\partial^2 g}{\partial t^2} + \frac{\{w, z\}}{4(t')^2} f - \frac{1}{4(t')^2} \left[\left(\frac{w''}{w'} - \frac{t''}{t'} \right)^2 + \frac{2w''}{z - w} \right] g = 0 \end{aligned} \tag{65}$$

where

$$(66)\qquad \begin{gathered} t' = \frac{dt}{dz} = \frac{i\sqrt{w'}}{z-w}, \qquad \frac{t''}{t'} = -\frac{1-w'}{z-w} + \frac{1}{2}\frac{w''}{w'}, \\ \frac{w''}{w'} - \frac{t''}{t'} = \frac{1-w'}{z-w} + \frac{1}{2}\frac{w''}{w'}. \end{gathered}$$

Finally we find, from (65),

$$(67)\qquad \begin{aligned} f_{\alpha\alpha} &= af + bg + cf_\alpha + dg_\beta, \\ g_{\beta\beta} &= a'f + b'g + c'f_\alpha + d'g_\beta, \end{aligned}$$

where

$$(68)\qquad \begin{gathered} a = 4 + n^2 - m^2, \qquad b = \frac{1}{4}\left[\frac{(z-w)^2}{w'}\{w,z\} + \frac{(z_0-w_0)^2}{w_0'}\{w_0,z_0\}\right] = -m_\beta, \\ c = 0, \qquad d = m, \\ a' = -b, \qquad b' = -a, \qquad c' = n, \qquad d' = 0. \end{gathered}$$

Equations (61) and (67) constitute the system of partial differential equations of our congruence, in its canonical form. The second form, given here for the value of b, is an immediate consequence of the following relations which are useful in a large number of questions;

$$(69)\qquad \begin{aligned} m_z &= -\frac{1}{4}\frac{z-w}{\sqrt{w'}}\{w,z\}, & m_{z_0} &= -\frac{1}{4}\frac{z_0-w_0}{\sqrt{w_0'}}\{w_0,z_0\}, \\ n_z &= \frac{i}{4}\frac{z-w}{\sqrt{w'}}\{w,z\}, & n_{z_0} &= -\frac{i}{4}\frac{z_0-w_0}{\sqrt{w_0'}}\{w_0,z_0\}, \\ m_t &= +\frac{i}{4}\frac{(z-w)^2}{w'}\{w,z\}, & m_{t_0} &= -\frac{i}{4}\frac{(z_0-w_0)^2}{w_0'}\{w_0,z_0\}, \\ n_t &= +\frac{1}{4}\frac{(z-w)^2}{w'}\{w,z\}, & n_{t_0} &= +\frac{1}{4}\frac{(z_0-w_0)^2}{w_0'}\{w_0,z_0\}, \end{aligned}$$

$$m_\alpha = \frac{i}{4}\left[\frac{(z-w)^2}{w'}\{w,z\} - \frac{(z_0-w_0)^2}{w_0'}\{w_0,z_0\}\right],$$

$$m_\beta = -\frac{1}{4}\left[\frac{(z-w)^2}{w'}\{w,z\} + \frac{(z_0-w_0)^2}{w_0'}\{w_0,z_0\}\right],$$

$$n_\alpha = -m_\beta, \qquad n_\beta = m_\alpha,$$

where the last two equations are merely expressive of the fact, already noted, that $m + in$ is a function of $x + iy$ and therefore of $\alpha + i\beta$.

The focal sheets of the congruence can degenerate into curves only if m or n, or both m and n, are equal to zero. Equations (69) show that this can happen only if w is a linear function of z, say

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \tag{70}$$

where we may assume $\alpha\delta - \beta\gamma = 1$, since the case $w = \text{const.}$ excluded by this hypothesis, is of no interest. We find from (62), if we choose

$$\sqrt{w'} = (\gamma z + \delta)^{-1}, \qquad m + in = -(\alpha + \delta), \qquad m - in = -(\alpha_0 + \delta_0),$$

so that

$$2m = -(\alpha + \alpha_0 + \delta + \delta_0), \qquad 2in = -(\alpha - \alpha_0 + \delta - \delta_0).$$

Consequently we obtain the following result.

The sheet S_f *of the focal surface is degenerate if and only if* w *is a linear function of* z *of the form*

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha\delta - \beta\gamma = 1,$$

for which the real parts of α *and* δ *are numerically equal but opposite in sign. The second sheet* S_g *degenerates if and only if the corresponding relation holds between the imaginary parts of* α *and* δ, *that is, if* α *and* δ *are conjugate complex.** *Both sheets degenerate if and only if*

$$\alpha + \delta = 0.$$

In all other cases, even if w is a linear function of z,† the focal surfaces are non-degenerate. Equations (68) show that

$$W = mn - c'd = 0$$

in all cases. Consequently *the congruence is a W-congruence,*‡ *that is, it has the property of making the asymptotic lines of the two sheets of the focal surface correspond to each other.* That this is so, also follows directly from equations (55). For we may regard (55) as the differential equations of a ruled surface $z_0 = \text{const.}$ of our congruence. The form of the equations shows immediately that the curves of contact of this ruled surface with the two sheets of the focal surface are asymptotic curves upon both of them.§ Of course the same facts may be read off from equations (64) or (65).

These simple remarks enable us to conclude further that the ruled surfaces

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* If besides, β and $-\gamma$ are conjugate complex, (70) reduces to a rotation of the Riemann sphere.

† If w is a linear function, aside from these exceptional cases, both sheets of the focal surface are quadrics.

‡ Brussels paper, p. 34 and p. 46.

§ E. J. Wilczynski, *Projective differential geometry of curves and ruled surfaces*, Leipzig (1906), p. 142.

z_0 = const. and z = const., that is, *the ruled surfaces of the congruence which are determined by the imaginary generators of the Riemann sphere are the ones which touch the focal surface along its asymptotic lines.*

Since these ruled surfaces are imaginary, while the focal surface is real, *both sheets of the focal surface must have positive curvature everywhere.* But the consideration of these ruled surfaces teaches us still more. The curve of contact with either sheet of the focal surface is an asymptotic curve, not only of the focal surface, but of the ruled surface as well. But this ruled surface has two straight line directrices, namely the two imaginary generators of the Riemann sphere which the points P_1 and P_2 describe when z_0 remains constant. Every asymptotic line on a ruled surface with two straight line directrices has the property that each of its asymptotic lines belongs to a linear complex, that is, there exists a linear complex which contains all of its tangents. It follows then that *every asymptotic line on each sheet of the focal surface of our congruence belongs to a linear complex.*

We wish to verify this conclusion analytically, and also to study some other questions which arise in this connection. If we eliminate g and g_β from equations (61) and (67) we find

$$\begin{aligned} f_{\alpha\alpha} - f_{\beta\beta} &= -2\frac{m_\beta}{m} f_\beta + af, \\ f_{\alpha\beta} &= \frac{m_\alpha}{m} f_\beta + mnf, \end{aligned} \tag{71}$$

the differential equations of the focal sheet S_f referred to the curves α = const. and β = const., which are the curves along which the developables of the congruence touch S_f. The second of these equations merely states the familiar fact that the curves α = const. and β = const. form a conjugate system on S_f; the first equation of (71) shows that *this system of curves on S_f is isothermally conjugate.*

Let us transfrom (71) by introducing $t = \alpha + i\beta$ and $t_0 = \alpha - i\beta$ as independent variables. We find a system of differential equations of the form

$$\begin{aligned} \frac{\partial^2 f}{\partial t^2} + 2A\frac{\partial f}{\partial t} + 2B\frac{\partial f}{\partial t_0} + Cf = 0, \\ \frac{\partial^2 f}{\partial t_0^2} + 2A'\frac{\partial f}{\partial t} + 2B'\frac{\partial f}{\partial t_0} + C'f = 0, \end{aligned} \tag{72}$$

where

$$\begin{aligned} A &= -\frac{1}{2}\frac{\partial \log m}{\partial t}, & B &= +\frac{1}{2}\frac{\partial \log m}{\partial t}, & C &= \frac{1}{4}[(m + in)^2 - 4], \\ A' &= +\frac{1}{2}\frac{\partial \log m}{\partial t_0}, & B' &= -\frac{1}{2}\frac{\partial \log m}{\partial t_0}, & C' &= \frac{1}{4}[(m - in)^2 - 4]. \end{aligned} \tag{73}$$

Equations (72) are the differential equations of the surface S_f referred to its asymptotic lines.[*] From (62), (63), and (73) we find

$$\frac{\partial^2 \log A'}{\partial t \partial t_0} = \frac{\partial^2 \log B}{\partial t \partial t_0} = 4A'B. \tag{74}$$

But these equations constitute the analytic verification, which we were seeking; they show, according to C. T. Sullivan, that the asymptotic curves of S_f belong to linear complexes.[†] Sullivan has also noted that every osculating ruled surface of such a surface has two straight-line directrices, and that the locus of these directrices is a quadric surface, which he calls the directrix quadric.[‡] It is evident, from what precedes, that in our case *the directrix quadric associated in this way with each of the two sheets of the focal surface is the Riemann sphere.*

Let us consider an arbitrary point P of the surface S_f. The osculating linear complexes of the two asymptotic curves of S_f, which cross at P, have a linear congruence in common. The directrices of this congruence are called the directrices of the point P; one of them passes through P while the other lies in the plane tangent to S_f at P. The resulting two congruences have the property that their developables correspond to each other and to a common net of curves on S_f called its directrix curves.[§] For a system of form (72), the differential equation of the directrix curves is

$$BL\,dt^2 + 2M\,dt\,dt_0 - A'N\,dt_0^2 = 0,$$

where

$$L = -2A'(2A'BF + 2A'BB_{t_0} + BA'_{tt}) + BA'^2_t,$$

$$M = A'^2B^2\frac{\partial^2 \log(B/A')}{\partial t \partial t_0},$$

$$N = -2B(2A'BG + 2A'BA'_t + A'B_{t_0t_0}) + A'B^2_{t_0}.$$ [||]

In our case this differential equation reduces to

$$[4-(m+in)^2]\,dt^2 - [4-(m-in)^2]\,dt_0^2 = 0. \tag{75}$$

Consequently *the directrix curves on S_f form a conjugate system and its equations can be obtained by quadratures.* This property however is not peculiar to our case; it is common to all surfaces whose asymptotic lines belong to linear complexes.* But if we set up the differential equation for the directrix curves on S_g we obtain (75) over again. Consequently *the directrix curves of the two sheets of the focal surface correspond to each other.* This property is not known to be an obvious consequence of the fact that our congruence is a W-congruence. On account of this latter property, the system of curves on S_g which corresponds to the directrix curves of S_f must be a conjugate system, but it need not therefore be composed of the directrix curves of S_g.

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[*] E. J. Wilczynski, *Projective differential geometry of curved surfaces.* (First Memoir), these Transactions, vol. 8 (1907), p. 244.

[†] C. T. Sullivan, *Properties of surfaces whose asymptotic lines belong to linear complexes,* these Transactions, vol. 13 (1912), p. 175.

[‡] L. c., p. 185.

[§] E. J. Wilczynski, *Projective differential geometry of curved surfaces* (Second Memoir), these Transactions, vol. 9 (1908), p. 95.

[||] L. c., p. 116, equations (136) and (137).

The integrals of (75) are

$$\int \sqrt{4 - (m + in)^2}\, dt \pm \int \sqrt{4 - (m - in)^2}\, dt_0 = \text{const.};$$

but these equations are equivalent to those obtained by equating to constants the real and imaginary parts of the function

$$\int \sqrt{4 - (m + in)^2}\, dt.$$

If we substitute for $m + in$ and dt their values, we obtain the following result.

The directrix curves on both sheets of the focal surface correspond to the same orthogonal isothermal system of curves of the $\xi\eta$-plane or on the sphere, namely to the system obtained by equating to constants the real and imaginary parts of the function

$$i \int \sqrt{4w' - \left[1 + w' + \frac{z - w}{2} \frac{w''}{w'}\right]^2} \frac{dz}{z - w}. \tag{76}$$

From this it follows, in particular, that *these directrix curves are real.* The same thing follows from the differential equation of the directrix curves when written in the form

$$mnd\alpha^2 - ad\alpha d\beta - mnd\beta^2 = 0, \tag{77}$$

obtained from (75) by introducing the real variables α and β in place of t and t_0. For, the coefficients and variables in (77) are real, and the discriminant

$$a^2 + 4m^2 n^2 = (4 - m^2 + n^2)^2 + 4m^2 n^2 \tag{78}$$

is positive.

The osculating planes of the curves $\alpha = $ const. and $\beta = $ const. which meet at a point P_f of S_f intersect in a line through P_f which is called the *axis* of P_f. The axes of all points of S_f form a congruence called *axis congruence,* and the curves on S_f, which correspond to the developables of this congruence, are called its *axis curves.* The differential equation of the axis curves on S_f are

$$\left(mn - \frac{\partial^2 \log m}{\partial\alpha\partial\beta}\right) d\alpha^2 - \left(a - \frac{m_{\alpha\alpha}}{m}\right) d\alpha d\beta - mnd\beta^2 = 0.\dagger \tag{79}$$

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* Sullivan, l.c., p. 194.

† E. J. Wilczynski, *The general theory of congruences,* these Transactions, vol. 16 (1915), p. 316, equation (9).

The axis curves on S_g are given by

$$(80)\qquad -mn\,d\alpha^2 + \left(a + \frac{n_{\beta\beta}}{n}\right) d\alpha d\beta + \left(mn - \frac{\partial^2 \log n}{\partial\alpha\partial\beta}\right) d\beta^2 = 0.$$

The Laplace transformation associates with every point of S_f two points in the plane tangent to S_f at P_f, namely the first and minus first Laplace transforms of P_f. The line joining these points is called the *ray* of P_f, and the totality of the rays of all points of S_f is called its *ray congruence.* The *ray curves* of S_f, that is, the curves of S_f which correspond to the developables of the ray congruence are given by

$$(81)\qquad -mn\,d\alpha^2 + \left(a - \frac{m_{\alpha\alpha}}{m}\right) d\alpha d\beta + \left(mn - \frac{\partial^2 \log m}{\partial\alpha\partial\beta}\right) d\beta^2 = 0.^*$$

Similarly we find as differential equation of the ray curves on S_g, the following:

$$(82)\qquad \left(mn - \frac{\partial^2 \log n}{\partial\alpha\partial\beta}\right) d\alpha^2 - \left(a + \frac{n_{\beta\beta}}{n}\right) d\alpha d\beta - mn\,d\beta^2 = 0.$$

The asymptotic curves on S_f or S_g are given by

$$(83)\qquad dt\,dt_0 = d\alpha^2 + d\beta^2 = 0.$$

Consequently, a relation of the form

$$d\alpha_1\, d\alpha_2 + d\beta_1\, d\beta_2 = 0$$

between two tangents of S_f at P_f expresses the property of conjugacy. Clearly this relation exists between the surface tangents given by (79) and (81), and also between (80) and (82).

Therefore our congruence has the following additional property. *On each sheet of the focal surface, each of its axis curve tangents is conjugate to one of the ray-curve tangents.*

In order to study these relations a little more thoroughly we propose to find the equations of the directrices, axes, and rays of the points P_f and P_g, referred to a common coördinate system. The most convenient system of coördinates for this purpose is one whose fundamental tetrahedron is made up of the four points P_f, P_g, P_ρ, and P_σ, where P_ρ and P_σ are defined by

$$(84)\qquad \rho = f_\alpha - \frac{m_\alpha}{m} f, \qquad \sigma = g_\beta - \frac{n_\beta}{n} g;$$

P_ρ and P_σ are the first and minus first Laplace transforms of P_f and P_g with respect to the conjugate systems determined on S_f and S_g respectively by the developables of the congruence.

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* L.c., p. 318, equation (23).

By direct application of the formulæ of the general theory of congruences we find the following results.*

The equations of the axis of P_f are

$$(85) \qquad x_3 = 0, \qquad x_2 + \left(\frac{m_\beta}{m} - \frac{n_\beta}{n}\right)x_4 = 0;$$

the axis of P_g is given by

$$(86) \qquad x_4 = 0, \qquad x_1 - \left(\frac{m_\alpha}{m} - \frac{n_\alpha}{n}\right)x_3 = 0.$$

The ray of P_f has the equations

$$(87) \qquad x_1 = x_4 = 0,$$

and the ray P_g is given by

$$(88) \qquad x_2 = x_3 = 0.$$

To find the equations of the directrices of P_f and P_g referred to this coördinate system is not quite so easy. In the projective theory of surfaces, the equations of the directrices of P_f are referred to a coördinate system whose vertices are given by†

$$(89) \qquad \begin{aligned} &f, \qquad \rho_f = f_t - \frac{1}{2}\frac{m_t}{m}f, \qquad \sigma_f = f_{t_0} - \frac{1}{2}\frac{m_{t_0}}{m}f, \\ &\tau_f = f_{tt_0} - \frac{1}{2}\frac{m_{t_0}}{m}f_t - \frac{1}{2}\frac{m_t}{m}f_{t_0} + \left[\frac{1}{4}\frac{m_t\, m_{t_0}}{m^2} - \frac{1}{4}\left(\frac{m_t}{m}\right)_{t_0} - \frac{1}{4}\left(\frac{m_{t_0}}{m}\right)_t\right]f, \end{aligned}$$

and it becomes necessary to obtain the equations of transformation between the two systems, which may be called the local coördinate systems of the *surface* S_f and of the *congruence*. We find the following relations

$$(90) \qquad \begin{aligned} \rho_f &= \frac{1}{4m}(m_\alpha + i m_\beta)f - \frac{i}{2}mg + \frac{1}{2}\rho, \\ \sigma_f &= \frac{1}{4m}(m_\alpha - i m_\beta)f + \frac{i}{2}mg + \frac{1}{2}\rho, \\ \tau_f &= \left[1 + \frac{1}{4}(n^2 - m^2) - \frac{1}{16}\frac{m_\alpha^2}{m^2} + \frac{3}{16}\frac{m_\beta^2}{m^2}\right]f + \frac{1}{4}\left(2m\frac{n_\beta}{n} - m_\beta\right)g \\ &\qquad - \frac{1}{4}\frac{m_\alpha}{m}\rho + \frac{1}{2}m\sigma. \end{aligned}$$

Let x_1, x_2, x_3, x_4 be the coördinates of a point with respect to the system f, g, ρ, σ, so that the expression

$$x_1 f + x_2 g + x_3 \rho + x_4 \sigma$$

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* L. c., equations (5).

† First Memoir, p. 248, equations (40).

will furnish the homogeneous coördinates of the point referred to our fundamental homogeneous cartesian system when f, g, ρ, σ are replaced by f_t, g_t, ρ_t, σ_t, $(t = 1, 2, 3, 4)$. Similarly, let x_1', x_2', x_3', x_4' be the homogeneous coördinates of the same point referred to the local coördinate system of the surface S_f. Then the equation

$$\omega'(x_1 f + x_2 g + x_3 \rho + x_4 \sigma) = x_1' f + x_2' \rho_f + x_3' \sigma_f + x_4' \tau_f,$$

must be valid for all values of f, g, ρ, σ after ρ_f, σ_f, τ_f have been replaced by their values (90); in this equation ω' is an arbitrary factor of proportionality whose value is of no particular importance, on account of the homogeneity of our coördinate systems, and which may be given a special value so as to simplify the formulæ.

We obtain in this way the following equations of transformation:

$$\begin{aligned} \omega' x_1 &= x_1' + \frac{1}{4}\frac{m_\alpha + i m_\beta}{m} x_2' + \frac{1}{4}\frac{m_\alpha - i m_\beta}{m} x_3' \\ &\qquad + \left(1 + \frac{1}{4}n^2 - \frac{1}{4}m^2 - \frac{1}{16}\frac{m_\alpha^2}{m^2} + \frac{3}{16}\frac{m_\beta^2}{m^2}\right) x_4', \\ \omega' x_2 &= -\frac{im}{2} x_2' + \frac{im}{2} x_3' + \frac{1}{4n}(2mn_\beta - nm_\beta) x_4', \\ \omega' x_3 &= \frac{1}{2} x_2' + \frac{1}{2} x_3' - \frac{1}{4}\frac{m_\alpha}{m} x_4', \\ \omega' x_4 &= \frac{1}{2} m x_4'. \end{aligned} \tag{91}$$

In computing the inverse of this transformation it is convenient to put $\omega' = 2m$. We find

$$\begin{aligned} x_1' &= 2mx_1 + \frac{m_\beta}{m} x_2 - m_\alpha x_3 + \left(m^2 - n^2 - 4 - \frac{1}{4}\frac{m_\alpha^2 + m_\beta^2}{m^2} - \frac{m_\beta n_\beta}{mn}\right) x_4, \\ x_2' &= 2ix_2 + 2mx_3 + \left(\frac{m_\alpha + i m_\beta}{m} - 2i\frac{n_\beta}{n}\right) x_4, \\ x_3' &= -2ix_2 + 2mx_3 + \left(\frac{m_\alpha - i m_\beta}{m} + 2i\frac{n_\beta}{n}\right) x_4, \\ x_4' &= 4x_4. \end{aligned} \tag{92}$$

Now, the equations of the directrix of the first kind of P_f are

$$x_4' = 0, \qquad x_1' + \frac{1}{2}\frac{A_t'}{A'} x_2' + \frac{1}{2}\frac{B_{t_0}}{B} x_3' = 0, \tag{93}$$

if we use the notations (73), and those for the directrix of the second kind are

$$x_2' + \frac{1}{2}\frac{B_{t_0}}{B}x_4' = 0, \qquad x_3' + \frac{1}{2}\frac{A_t'}{A'}x_4' = 0.^* \tag{94}$$

Making use of (92) we find

$$x_4 = 0, \qquad x_1 - \frac{m_\alpha}{m}x_3 = 0 \tag{95}$$

as equations of the directrix of the first kind of S_f, and

$$x_3 = 0, \qquad x_2 - \frac{n_\beta}{n}x_4 = 0 \tag{96}$$

for the directrix of the second kind of S_f, both referred to the local coördinate system of the congruence.

In order to obtain the corresponding equations for the directrices of P_g, we must supply some formulæ for the surface S_g which are analogous to certain corresponding formulæ for S_f, but which we have not noted so far. The equations of S_g referred to the curves α = const. and β = const. are

$$\begin{aligned} g_{\alpha\alpha} - g_{\beta\beta} &= 2\frac{n_\alpha}{n}g_\alpha + ag, \\ g_{\alpha\beta} &= mng + \frac{n_\beta}{n}g. \end{aligned} \tag{97}$$

The equations of S_g referred to its asymptotic curves are

$$\begin{aligned} g_{tt} - \frac{n_t}{n}g_t - \frac{n_t}{n}g_{t_0} + \frac{1}{4}[(m+in)^2 - 4]g &= 0, \\ g_{t_0t_0} - \frac{n_{t_0}}{n}g_t - \frac{n_{t_0}}{n}g_{t_0} + \frac{1}{4}[(m-in)^2 - 4]g &= 0. \end{aligned} \tag{98}$$

The fundamental local coördinate system of S_g is given by the four expressions

$$\begin{aligned} &g, \qquad \rho_g = g_t - \frac{1}{2}\frac{n_t}{n}g, \qquad \sigma_g = g_{t_0} - \frac{1}{2}\frac{n_{t_0}}{n}g, \\ &\tau_g = g_{tt_0} - \frac{1}{2}\frac{n_{t_0}}{n}g_t - \frac{1}{2}\frac{n_t}{n}g_{t_0} + \frac{3}{4}\frac{n_t n_{t_0}}{n^2}g, \end{aligned} \tag{99}$$

so that

$$\begin{aligned} \rho_g &= \frac{1}{2}nf - \frac{n_\alpha + in_\beta}{4n}g - \frac{i}{2}\sigma, \\ \sigma_g &= \frac{1}{2}nf - \frac{n_\alpha - in_\beta}{4n}g + \frac{i}{2}\sigma, \end{aligned}$$

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* Second Memoir, p. 95, equations (70*a*) and (70*b*).

$$(100)\quad \tau_g = \frac{n}{4}\left(2\frac{m_\alpha}{m} - \frac{n_\alpha}{n}\right)f - \left(1 + \frac{1}{4}n^2 - \frac{1}{4}m^2 - \frac{3}{16}\frac{n_\alpha^2}{n^2} + \frac{1}{16}\frac{n_\beta^2}{n^2}\right)g + \frac{1}{2}n\rho - \frac{1}{4}\frac{n_\beta}{n}\sigma .$$

If the coördinates of a point, $x_1 f + x_2 g + x_3 \rho + x_4 \sigma$, referred to this system are x_1'', x_2'', x_3'', x_4'', we find

$$(101)\quad \begin{aligned} \omega'' x_1 &= \frac{1}{2}nx_2'' + \frac{1}{2}nx_3'' + \frac{n}{4}\left(2\frac{m_\alpha}{m} - \frac{n_\alpha}{n}\right)x_4'', \\ \omega'' x_2 &= x_1'' - \frac{n_\alpha + in_\beta}{n}x_2'' - \frac{n_\alpha - in_\beta}{n}x_3'' \\ &\qquad - \left(1 + \frac{1}{4}n^2 - \frac{1}{4}m^2 - \frac{3}{16}\frac{n_\alpha^2}{n^2} + \frac{1}{16}\frac{n_\beta^2}{n^2}\right)x_4'', \\ \omega'' x_3 &= \frac{1}{2}nx_4'', \\ \omega'' x_4 &= \frac{-i}{2}x_2'' + \frac{i}{2}x_3'' - \frac{1}{4}\frac{n_\beta}{n}x_4'', \end{aligned}$$

whence

$$(102)\quad \begin{aligned} x_1'' &= \frac{n_\alpha}{n}x_1 + 2nx_2 + \left(4 + n^2 - m^2 - \frac{n_\alpha^2 + n_\beta^2}{4n^2} - \frac{m_\alpha n_\alpha}{mn}\right)x_3 - n_\beta x_4, \\ x_2'' &= 2x_1 + \left(\frac{n_\alpha + in_\beta}{n} - 2\frac{m_\alpha}{m}\right)x_3 + 2nix_4, \\ x_3'' &= 2x_1 + \left(\frac{n_\alpha - in_\beta}{n} - 2\frac{m_\alpha}{m}\right)x_3 - 2nix_4, \\ x_4'' &= 4x_3 . \end{aligned}$$

From these formulæ we find

$$(103)\quad x_3 = 0, \qquad x_2 - \frac{n_\beta}{n}x_4 = 0$$

as the equations of the directrix of the first kind of P_g, and

$$(104)\quad x_4 = 0, \qquad x_1 - \frac{m_\alpha}{m}x_3 = 0$$

for the directrix of the second kind of P_g. Comparison with (95) and (96) furnishes the following theorem:

The two focal sheets of our congruence are so related that the directrix of the first kind belonging to any point of either sheet is at the same time the directrix of the second kind of the corresponding point of the other sheet.

THE UNIVERSITY OF CHICAGO,
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